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Upper–lower class tests and frequency results along subsequences

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Abstract

We characterize, by means of an integral test, the upper and lower classes associated with the law of the iterated logarithm for subsequences. We also prove strong laws for the frequency of indices in connection with the lower class.

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1. Introduction and results

Let $X = \{X_i, i \geqslant 1\}$ be a sequence of independent, identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$ and set $S_n(X) = S_n = X_1 + \cdots + X_n$, $n \geqslant 1$. The classical law of the iterated logarithm tells us that

$$\mathbf{P}\left\{ \limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \right\} = 1. \tag{1.1}$$

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Under minor additional conditions the sequence (X_n) satisfies also the Kolmogorov–Erdős–Petrovski test: for any positive nondecreasing function $\{\varphi(n), n \geqslant 1\}$ the probability

$$\mathbf{P}\{S_n > \sqrt{n}\varphi(n) \text{ infinitely often}\} \tag{1.2}$$

equals zero or one, depending on whether

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \mathrm{e}^{-\varphi^2(n)/2} \tag{1.3}$$

converges or diverges. See e.g. Erdős [4], Feller [5,6]. Feller [6] showed that the test holds if $EX_1 = 0$, $EX_1^2 = 1$ and

$$E(X_1^2 I(X_1 \geqslant t)) = \mathrm{O}((\log\log t)^{-1}) \quad \text{as } t \to \infty \tag{1.4}$$

and this result is best possible. (See Einmahl [3] for further information on this point.) Relation (1.4) holds, e.g. if

$$E(X_1^2 \log_+ \log_+ |X_1|) < \infty. \tag{1.5}$$

As customary, we say that φ belongs to the upper (lower) class of X, and write $\varphi \in \mathscr{U}(X)$ (resp. $\varphi \in \mathscr{L}(X)$), when the probability in (1.2) equals 0 (resp. 1).

The purpose of the present paper is to study the LIL and the upper–lower class problem for S_n when n runs over some subsequence $\mathscr{N}$.

Concerning (1.1) first, the law of the iterated logarithm for subsequences has been characterized in Weber [18]. In order to state it, we recall the notation from Sections 2 and 3 of [18]. Let $\mathscr{N} = \{n_k, k \geqslant 1\}$ be any increasing sequence of integers and $M > 1$; the value of M will be irrelevant. Let $I_0 = [0, M)$ and for each integer $k \geqslant 1$, let $I_k(M) = I_k = [M^k, M^{k+1})$. The subsequence of intervals I_k such that $I_k \cap \mathscr{N} \neq \emptyset$, determines an increasing sequence of indices, which we denote by $\kappa = \{\kappa_p, p \geqslant 1\}$. For any $n \in \mathscr{N}$ we put

$$\varphi^*(n) = \varphi^*(\mathscr{N}, M, n) = \sqrt{2\log(p+2)} \quad \text{if } n \in \mathscr{N} \cap I_{\kappa_p}.$$

The corresponding property to (1.1) for arbitrary subsequences can then be stated as follows:

$$\mathbf{P}\left\{ \limsup_{n\to\infty,\, n \in \mathscr{N}} \frac{S_n}{\sqrt{n}\varphi^*(n)} = 1 \right\} = 1. \tag{1.6}$$

See Gut [7], Gut and Schwabe [8] for related results. For similar results along random subsequences, see Chang and Hsiung [1], Rychlik and Zygo [14], and references therein. For Banach space generalizations, see Weber [19]. For Strassen versions of (1.6), see Lifshits and Weber [11].

The contrast between (1.1) and (1.6) is significant when $\mathscr{N}$ increases sufficiently rapidly, like $n_k = 2^{2^k}$, in which case $\varphi^*(n) \sim \sqrt{2\log\log\log n}$. A link between (1.1) and (1.6) can, however, be obtained through some information concerning the distribution of the sequence $\mathcal{N}$. Define

$$\Lambda(\mathcal{N}) = \limsup_{j\to\infty} \left\{ \frac{\log \#(i \leqslant j : \mathcal{N} \cap [M^i, M^{i+1}) \neq \emptyset)}{\log j} \right\}^{1/2}.$$

Then

$$\mathbf{P}\left\{ \limsup_{n\to\infty,\, n\in\mathcal{N}} \frac{S_n}{\sqrt{2n \log\log n}} = \Lambda(\mathcal{N}) \right\} = 1. \tag{1.7}$$

An improvement of the lower half of (1.6) was recently obtained in Weber [20, Theorem 6.1]. Let $0 \leqslant c < 1$; then

$$\mathbf{P}\left\{ \lim_{j\to\infty} \frac{\log \#(i \leqslant j : \exists n \in \mathcal{N} \cap [M^i, M^{i+1}) : S_n > c\sqrt{n}\varphi^*(n))}{\log \#(i \leqslant j : \mathcal{N} \cap [M^i, M^{i+1}) \neq \emptyset)} = 1 - c^2 \right\} = 1. \tag{1.8}$$

The case $c = 1$ remained open; this will be settled and extended further in Section 4.

We define now the upper and lower classes (near infinity) of X, relatively to an arbitrary increasing sequence of positive integers $\mathcal{N} = \{n_k, k \geqslant 1\}$. Let $\varphi = \{\varphi(n), n \geqslant 1\}$ be a nondecreasing sequence of positive reals. We say that $\varphi \in \mathcal{L}_{\mathcal{N}}(X)$, (resp. $\varphi \in \mathcal{U}_{\mathcal{N}}(X)$), when

$$\mathbf{P}\{S_n(X) > \sqrt{n}\varphi(n) \ \ n \in \mathcal{N}, \text{ infinitely often}\} = 1 \quad (\text{resp.} = 0). \tag{1.9}$$

A necessary preliminary (but classical) step in dealing with our problem will be the study of the upper and lower classes of Brownian motion $W = \{W_t, t \geqslant 0\}$. The classes near infinity for Brownian motion, $\mathcal{L}_{\mathcal{N}}(W)$ (resp. $\mathcal{U}_{\mathcal{N}}(W)$), are defined as above, just with $X_1 \stackrel{\mathcal{L}}{=} \mathcal{N}(0, 1)$.

Before going further, we shall make some useful observations. As is well-known in the study of the law of the iterated logarithm along the whole sequence of integers, it is enough to control the behavior of partial sums along a geometric subsequence, say $\mathcal{N}_0 = \{2^k, k = 1, 2, \ldots\}$. In other words, we sieve the integers with a *global sieve* represented by the intervals $[2^k, 2^{k+1}), k = 1, 2, \ldots$. The same rule operates to characterize the law of the iterated logarithm for subsequences. For characterizing upper and lower classes for subsequences, the crucial new fact is the appearance of a *local sieve* of the subsequence $\mathcal{N}$, which will be also used to express the characterization. More precisely, endow $\mathbf{R}^+$ with the metric induced by the normalized Brownian motion

$$d(s, t) = \| W_s/\sqrt{s} - W_t/\sqrt{t} \|_2 = [2(1 - \sqrt{s/t})]^{1/2}, \quad s, t \in \mathbf{R}^+, \ s \leqslant t. \tag{1.10}$$

For any subset T of $\mathbf{R}^+$ and any real $u > 0$, define by $M(T, u) = M_d(T, u)$ the maximal number of points of T which are mutually at distance $\geqslant u$ (equal to 1 if none); and by $N(T, u) = N_d(T, u)$ the minimal covering number (possibly infinite) of T by d-open balls of radius u centered in T. One has the immediate relations $M(T, 2u) \leqslant N(T, u) \leqslant M(T, u)$. Let $M > 1$ be fixed. The following characterization is established in Sections 2 and 3.

Theorem 1.1. *Let $X=\{X_i, i\geqslant 1\}$ be an i.i.d. sequence satisfying $EX_1=0$, $EX_1^2=1$ and* (1.5). *Then for any increasing sequence $\mathcal{N}=\{n_k, k\geqslant 1\}$ of positive integers and any nondecreasing sequence $\{\varphi(n), n\geqslant 1\}$ of positive reals, the following assertions are equivalent*:

(i) $\varphi \in \mathcal{U}_{\mathcal{N}}(X)$, (*resp.* $\mathcal{L}_{\mathcal{N}}(X)$)

(ii) $$\sum_{p=1}^{\infty} \frac{1}{\hat\varphi_p} \exp(-\hat\varphi_p^2/2) M(\mathcal{N}\cap I_{\kappa_p}, 1/\hat\varphi_p) < \infty, \quad (resp. = \infty). \tag{1.11}$$

Here $\hat\varphi_p$ denotes the value of φ at the smallest element of $\mathcal{N}\cap I_{\kappa_p}$.

The quantity $M(\mathcal{N}\cap I_{\kappa_p}, 1/\hat\varphi_p)$ relates to a notion of local packing for $\mathcal{N}$. It defines the maximal number of points of $\mathcal{N}\cap I_{\kappa_p}$ which are mutually at distance $\geqslant 1/\hat\varphi_p$ for the metric induced by the normalized Brownian motion.

In the case when $\mathcal{N}$ is the whole sequence of positive integers, Theorem 1.1 reduces to the Kolmogorov–Erdős–Petrovski test (see the calculations below), and as shown in Einmahl [3], Feller [6], the moment assumption (1.5) is nearly optimal here.

The test sum in (1.11) can be written in another form, convenient for applications. Assume that

$$c_1(\log p)^{1/2} \leqslant \hat\varphi_p \leqslant c_2(\log p)^{1/2} \tag{1.12}$$

for some positive constants c_1, c_2. As we will see in Section 2, there is no loss of generality in assuming (1.12). Let $r=[\log p]$ and let $u_0<u_1<\cdots<u_r$ be a geometric sequence with $u_0=M^{\kappa_p}$, $u_r=M^{\kappa_p+1}$. In view of (1.10), $\{u_0,\dots,u_r\}$ is a d-equidistant set in $[M^{\kappa_p}, M^{\kappa_p+1}]$, which divides this interval into $[\log p]$ subintervals $J_1^{(p)},\dots,J_r^{(p)}$ of equal d-length

$$[2(1-M^{-1/2r})]^{1/2} \sim (\log M)^{1/2}(\log p)^{-1/2}.$$

Let N_p denote the number of intervals $J_\nu^{(p)}$ containing at least one point of $\mathcal{N}$. By (1.12) the length of the intervals $J_\nu^{(p)}$ is within constant multiples of $1/\hat\varphi_p$ and thus

$$c_3 N_p \leqslant M(\mathcal{N}\cap I_{\kappa_p}, 1/\hat\varphi_p) \leqslant c_4 N_p \tag{1.13}$$

with suitable constants c_3, c_4. Hence the sum in (1.11) is equiconvergent with

$$\sum_{p=1}^{\infty} \frac{1}{\hat\varphi_p} \exp(-\hat\varphi_p^2/2) N_p. \tag{1.14}$$

Clearly, $1\leqslant N_p \leqslant \log p \leqslant \text{const}\, \hat\varphi_p^2$. If $\mathcal{N}$ contains exactly one element in each $[M^{\kappa_p}, M^{\kappa_p+1}]$ (e.g. if $n_k=M^k$), then (1.14) reduces to $\sum_{p=1}^{\infty} \frac{1}{\hat\varphi_p} \exp(-\hat\varphi_p^2/2)$, while if $\mathcal{N}$ intersects all intervals $J_\nu^{(p)}$, then (1.14) is equiconvergent with

$$\sum_{p=1}^{\infty} \hat\varphi_p \exp(-\hat\varphi_p^2/2). \tag{1.15}$$

We thus see the interesting fact that once $\mathcal{N}$ contains an element in each $J_\nu^{(p)}$, the upper–lower class character of a function φ remains the same if we add new elements to $\mathcal{N}$; in particular, it is the same as in the case when $\mathcal{N}$ is the whole sequence of integers. In this case $\hat{\varphi}_p = \varphi(M^p)$ and (1.15) becomes

$$\sum_{p=1}^{\infty} \varphi(M^p) \exp\{-\varphi^2(M^p)/2\}$$

which is easily seen to be equiconvergent with the sum

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \exp\{-\varphi^2(n)/2\}$$

appearing in the Kolmogorov–Erdős–Petrovski test. Finally, we observe that if

$$\hat{\varphi}_p = c(2\log p)^{1/2}, \quad (c>0)$$

then the sum (1.14) will converge for $c>1$ and diverge for $c\leqslant 1$, regardless the value of $1\leqslant N_p\leqslant \log p$. This yields (1.6) and shows that the "crude" LIL behavior of S_n along $\mathcal{N}$ does not depend on how many elements $\mathcal{N}$ has in the intervals $J_\nu^{(p)}$. On the other hand, if $\hat{\varphi}_p = (2\log p + c\log\log p)^{1/2}$, then φ belongs to the upper or lower class according as

$$\sum_{p=1}^{\infty} \frac{N_p}{p(\log p)^{(1+c)/2}}$$

converges or diverges. Thus in this first refinement of the LIL (1.6) the structure of $\mathcal{N}$ becomes important. Specifically, if $N_p \sim (\log p)^\alpha$ $(0<\alpha<1)$, then φ belongs to the upper class iff $c>1+2\alpha$.

If condition (1.12) is not satisfied, our previous considerations remain still valid, provided we choose $r=[\hat{\varphi}_p^2]$ instead of $r=[\log p]$. However, in this case the intervals $J_\nu^{(p)}$ will depend on φ. Similarly as above, we get that the d-length of the intervals $J_\nu^{(p)}$ is $\sim \text{const}\,\hat{\varphi}_p^{-1}$ and thus

$$M(\mathcal{N}\cap I_{\kappa_p}, 1/\hat{\varphi}_p)\leqslant \text{const}\,\hat{\varphi}_p^2. \tag{1.16}$$

2. Upper and lower classes for partially observed Brownian motion

In this section we prove Theorem 1.1 in the Brownian case, i.e. when $X_1 \stackrel{\mathcal{L}}{=} \mathcal{N}(0,1)$. Let $W=\{W_t, t\geqslant 0\}$ be a linear Brownian motion and put

$$\Psi(x) = 1-\Phi(x) = (2\pi)^{-1/2}\int_x^\infty \mathrm{e}^{-t^2/2}\,\mathrm{d}t, \quad x\in\mathbf{R}.$$

As is well known,

$$\Psi(x) \sim (2\pi)^{-1/2}x^{-1}\mathrm{e}^{-x^2/2} \quad \text{as } x\to\infty. \tag{2.1}$$

Let $1 \leqslant a < b < \infty$, and $T \subset [a,b]$. The entropy numbers $M(T,u)$, $N(T,u)$, $u>0$, associated to T are those defined above with respect to the metric d in (1.10). As we already noted, $N(T,u) \leqslant M(T,u) \leqslant N(T,u/2)$ for any $u>0$. Also, if u is small and $0<x<y<z$ satisfy $d(x,y)=d(y,z)=u$, then $d(x,z) \sim u\sqrt{2}$, whence it follows that for sufficiently small $u>0$ we have

$$M(T,u) \leqslant M(T,u/2) \leqslant 8M(T,u). \tag{2.2}$$

Proposition 2.1. *There exist positive constants C_1, C_2, C_3, depending on b/a only, such that for any real $x \geqslant C_3$ we have*

$$C_1 \leqslant \frac{\mathbf{P}\{\sup_{t\in T} W_t/\sqrt{t} > x\}}{\Psi(x)M(T,1/x)} \leqslant C_2.$$

The above proposition is obtained as a straightforward application of Theorem 2.2 of Weber [17], which we are going to recall for convenience. Here T is temporarily an arbitrary parameter set. Put, for any centered Gaussian process $X = \{X_t(\omega), \omega \in \Omega, t \in T\}$,

$$m_X(\varepsilon) = \sup\{\mathbf{E}[\sup\{X_u : \|X_u - X_t\|_{L^2(\mathbf{P})} \leqslant \varepsilon\}], t \in T\},$$

and note in what follows: $d_X(s,t) = \|X_s - X_t\|_{L^2(\mathbf{P})}$, $s,t \in T$.

Lemma 2.2. *Assume that X has sample bounded paths and that $\mathbf{E}X_t^2 = 1$ for any $t \in T$. Fix $0<\gamma<1$ and $0<\xi\leqslant\frac{1}{4}$, $H>1$ such that $\exp\{-H^2(1-\xi)\}<\gamma/2$ and put*

$$\varepsilon(x) = \inf\left\{\varepsilon>0 : \sup\left\{\frac{2}{\xi}\left[\sup_{\substack{k\varepsilon\leqslant \mathrm{diam}(T,d_X)\\ k\ \text{integer}}} \frac{k_0 m_X(k\varepsilon)}{k\varepsilon^2}\right], \frac{H}{\varepsilon}\right\} \leqslant \frac{x}{2\sqrt{2}}\right\},$$

where k_0 is some universal constant (arising from Chevet–Sudakov inequality). Then, for all $x \geqslant 0$

$$(1-\gamma)M_{d_X}[T,\varepsilon(x)]\Psi(x) \leqslant \mathbf{P}\left\{\sup_{t\in T} X_t > x\right\},$$

$$\mathbf{P}\left\{\sup_{t\in T} X_t > x + 2m_X[\varepsilon(x)]\right\} \leqslant M_{d_X}[T,\varepsilon(x)]\Psi(x).$$

(In the statement in Weber [17], there is another parameter: α which turns out to be equal to one because the variance is constant.)

Proof of Proposition 2.1. Put $X = \{X_t = W_t/\sqrt{t}, t \geqslant 0\}$, and for any $\theta \in T$, $r>0$: $\mathscr{B}(\theta,r) = \{t \in T : \|X_\theta - X_t\|_2 < r\}$. Let $s,t \in T$, $s \leqslant t$; since $\|X_s - X_t\|_2^2 = 2(1-\sqrt{s/t}) = 2\frac{t-s}{\sqrt{t}(\sqrt{t}+\sqrt{s})}$, one has

$$\frac{1}{\sqrt{b}} \leqslant \frac{\|X_s - X_t\|_2}{\sqrt{|s-t|}} \leqslant \frac{1}{\sqrt{a}}.$$

Let $\theta \in T$ and $\varepsilon > 0$ such that $(\theta - \varepsilon^2, \theta + \varepsilon^2) \subset [a, b]$. Let $\varepsilon_1 = \varepsilon/\sqrt{a}$. Then

$$T \cap (\theta - \varepsilon_1^2 a, \theta + \varepsilon_1^2 a) \subset \mathscr{B}(\theta, \varepsilon_1) \subset T \cap (\theta - \varepsilon_1^2 b, \theta + \varepsilon_1^2 b).$$

Let $0 < v_1 \leqslant \varepsilon_1$. We put $v = v_1\sqrt{a}$. We consider a subdivision of $(\theta - \varepsilon_1^2 b, \theta + \varepsilon_1^2 b)$ of size v^2. Its length is bounded by

$$\left[\frac{2\varepsilon_1^2 b}{v^2}\right] + 1 \leqslant 4\left(\frac{\varepsilon_1^2 b}{v^2}\right).$$

Let I be an arbitrary interval of the subdivision. If $I \cap T \neq \emptyset$, pick some point in $I \cap T$ which we denote by t_I. To any $t \in T$, there corresponds an interval I such that $t \in I$, and since $I \subset (t_I - v^2, t_I + v^2)$, we deduce

$$\begin{aligned} \mathscr{B}(\theta, \varepsilon_1) &\subset T \cap (\theta - \varepsilon_1^2 b, \theta + \varepsilon_1^2 b) \\ &= \sum_{I: I \cap T \neq \emptyset} I \cap T \subset \sum_{I: I \cap T \neq \emptyset} (t_I - v^2, t_I + v^2) \cap T \\ &\subset \sum_{I: I \cap T \neq \emptyset} (t_I - v^2, t_I + v^2) \cap T \subset \sum_{I: I \cap T \neq \emptyset} \mathscr{B}(t_I, v_1). \end{aligned}$$

We thus have for any $0 < v_1 \leqslant \varepsilon_1$

$$N(\mathscr{B}(\theta, \varepsilon_1), v_1) \leqslant 4\left(\frac{\varepsilon_1^2 b}{v^2}\right) = 4\left(\frac{\varepsilon_1^2 b}{v_1^2 a}\right).$$

We can now estimate $\mathbf{E}\sup_{t \in \mathscr{B}(\theta, \varepsilon_1)} X_t = \frac{1}{2}\mathbf{E}\sup_{s,t \in \mathscr{B}(\theta, \varepsilon_1)} (X_s - X_t)$. We have classically

$$\mathbf{E} \sup_{s,t \in \mathscr{B}(\theta, \varepsilon_1)} (X_s - X_t) \leqslant K \int_0^{\operatorname{diam}(\mathscr{B}(\theta, \varepsilon_1))} \sqrt{\log N(\mathscr{B}(\theta, \varepsilon_1), v_1)}\, \mathrm{d}v_1,$$

where K is some universal constant. So

$$\begin{aligned} &\int_0^{\operatorname{diam}(\mathscr{B}(\theta, \varepsilon_1))} \sqrt{\log N(\mathscr{B}(\theta, \varepsilon_1), v_1)}\, \mathrm{d}v_1 \\ &\quad \leqslant \int_0^{2\varepsilon_1} \sqrt{\log 4\left(\frac{\varepsilon_1^2 b}{v_1^2 a}\right)}\, \mathrm{d}v_1 \overset{v_1 = h\varepsilon_1}{=} \varepsilon_1 \int_0^2 \sqrt{\log\left(\frac{4b}{ah^2}\right)}\, \mathrm{d}h. \end{aligned}$$

Hence

$$\mathbf{E} \sup_{t \in \mathscr{B}(\theta, \varepsilon_1)} X_t \leqslant \operatorname{Const}(a, b)\varepsilon_1,$$

where $\operatorname{Const}(a, b) = \frac{1}{2}K\int_0^2 \sqrt{\log(4b/ah^2)}\, \mathrm{d}h$. Therefore, recalling that

$$m_X(\varepsilon) = \sup\{\mathbf{E}[\sup\{X_u : \|X_u - X_t\|_{L^2(\mathbf{P})} \leqslant \varepsilon\}], t \in T\},$$

we see that $m_X(\varepsilon) \leqslant \operatorname{Const}(a, b)\varepsilon$; hence

$$\sup_{\substack{k\varepsilon \leqslant \operatorname{diam}(T,d) \\ k \text{ integer}}} \frac{k_0 m_X(k\varepsilon)}{k\varepsilon^2} \leqslant \frac{k_0 \operatorname{Const}(a, b)}{\varepsilon}.$$

Choosing now H so that $H > H(a,b,\xi) := 2k_0 \operatorname{Const}(a,b)/\xi$, we also find that

$$\sup\left\{\frac{2}{\xi}\left[\sup_{\substack{k\varepsilon\leqslant \operatorname{diam}(T,d)\\ k \text{ integer}}} \frac{k_0 m_X(k\varepsilon)}{k\varepsilon^2}\right], \frac{H}{\varepsilon}\right\} = \frac{H}{\varepsilon}.$$

Thus,

$$\varepsilon(x) = \inf\left\{\varepsilon > 0 : \frac{H}{\varepsilon} \leqslant \frac{x}{2\sqrt{2}}\right\} = \frac{2\sqrt{2}H}{x}$$

and

$$2m_X[\varepsilon(x)] \leqslant 2\operatorname{Const}(a,b)\varepsilon(x) = \frac{4\sqrt{2}\operatorname{Const}(a,b)H}{x}.$$

From Lemma 2.2 we get

$$\begin{aligned}\mathbf{P}\left\{\sup_{t\in T} X_t > x + \frac{4\sqrt{2}\operatorname{Const}(a,b)H}{x}\right\} &\leqslant \mathbf{P}\left\{\sup_{t\in T} X_t > x + 2m_X[\varepsilon(x)]\right\}\\ &\leqslant M[T,\varepsilon(x)]\Psi(x)\end{aligned}$$

for any $x > 0$, hence letting $c_0 = 4\sqrt{2}\operatorname{Const}(a,b)H$ and noting also that the function $y = x + c_0/x$ is increasing for $x \geqslant \sqrt{c_0}$ and maps $[\sqrt{c_0}, \infty)$ onto $[2\sqrt{c_0}, \infty)$ and that the function $\Psi(x + c_0/x)/\Psi(x)$ remains between positive bounds for $x \geqslant \sqrt{c_0}$, it follows that

$$\mathbf{P}\left\{\sup_{t\in T} X_t > x\right\} \leqslant C_4 M[T,\varepsilon(x)]\Psi(x), \quad \text{for } x \geqslant C_4,$$

where C_4 is a constant depending on H and b/a. Another application of Lemma 2.2 yields, choosing $\gamma = \frac{1}{2}$,

$$\mathbf{P}\left\{\sup_{t\in T} X_t > x\right\} \geqslant \frac{1}{2} M[T,\varepsilon(x)]\Psi(x).$$

The proof of Proposition 2.1 is now complete, upon choosing $H(a,b) = H(a,b,\frac{1}{4})$ and using (2.2). $\square$

We now pass to the

Proof of Theorem 1.1 in the Brownian case. We first show that without loss of generality we can assume that φ is constant on each $I_{\kappa_p} \cap \mathcal{N}$. To this purpose, let n_p^* resp. n_p^{**} denote the smallest, resp. largest element of $I_{\kappa_p} \cap \mathcal{N}$ and let $H \subset N$ denote the set of integers $p \geqslant 1$ for which the increment of $\varphi^2(n)$ over $I_{\kappa_p} \cap \mathcal{N}$ is at least 1, i.e. $\varphi^2(n_p^{**}) - \varphi^2(n_p^*) \geqslant 1$. If p and q belong to H and $p < q$, then $n_q^* \geqslant n_p^{**}$ and thus $\varphi^2(n_q^*) - \varphi^2(n_p^*) \geqslant \varphi^2(n_p^{**}) - \varphi^2(n_p^*) \geqslant 1$, whence

$$\exp(\varphi^2(n_q^*)/4 - \varphi^2(n_p^*)/4) \geqslant \mathrm{e}^{1/4}.$$

Thus $\exp(\varphi^2(n_p^*)/4)$, $p \in H$ grows at least exponentially and thus using (1.16) we get

$$\sum_{p\in H} \frac{1}{\varphi(n_p^*)} \mathrm{e}^{-\varphi^2(n_p^*)/2} M(I_{\kappa_p} \cap \mathcal{N}, 1/\varphi(n_p^*)) \leqslant C \sum_{p\in H} \varphi(n_p^*)\mathrm{e}^{-\varphi^2(n_p^*)/2}$$
$$\leqslant C' \sum_{p\in H} \mathrm{e}^{-\varphi^2(n_p^*)/4} < \infty.$$

Hence by Proposition 2.1 we also have

$$\sum_{p\in H} \mathbf{P}\left(\sup_{t\in I_{\kappa_p}\cap\mathcal{N}} W_t/\sqrt{t} > \varphi(n_p^*)\right) < +\infty.$$

Hence, in view of the Borel–Cantelli lemma, discarding the sets $I_{\kappa_p} \cap \mathcal{N}$, $p \in H$ from the set $\mathcal{N}$ changes neither the convergence character of the sum

$$\sum_{p=1}^{\infty} \frac{1}{\varphi(n_p^*)} \mathrm{e}^{-\varphi^2(n_p^*)/2} M(I_{\kappa_p} \cap \mathcal{N}, 1/\varphi(n_p^*))$$

nor the upper–lower class character of $\{\varphi(n), n \in \mathcal{N}\}$. Thus without loss of generality we can assume that $\varphi^2(n)$ changes by at most 1 on every set $I_{\kappa_p} \cap \mathcal{N}$, but then using (2.2) we see that the convergence character of the last sum does not change if we replace n_p^* by n_p^{**}. Thus letting φ_{L}, resp. φ_{U} denote the functions obtained by replacing φ by its smallest, resp. largest value in each interval $I_{\kappa_p} \cap \mathcal{N}$, the series (1.11) belonging to φ_{L} and φ_{U} are equiconvergent and thus if we assume that Theorem 1.1 is valid under the above piecewise constancy assumption on φ, then we see that φ_{L} and φ_{U} belong to the same class relative to $\mathcal{N}$. But $\varphi_{\mathrm{L}} \leqslant \varphi \leqslant \varphi_{\mathrm{U}}$ and thus φ belongs to the same class as φ_{L} and φ_{U}.

Let us now assume that on each interval $I_{\kappa_p} \cap \mathcal{N}$ the function φ takes a constant value, which we will denote, without danger of confusion, by φ_p, instead of $\hat{\varphi}_p$. Let

$$Z_p = \sup_{t\in I_{\kappa_p}\cap\mathcal{N}} W_t/\sqrt{t}, \quad M_p = M(\mathcal{N} \cap I_{\kappa_p}, 1/\varphi_p).$$

Theorem 1.1 can then be restated as

$$P(Z_p \geqslant \varphi_p \text{ i.o.}) = 0 \quad \text{iff} \quad \sum_{p=1}^{\infty} \varphi_p^{-1}\mathrm{e}^{-\varphi_p^2/2} M_p < \infty. \tag{2.3}$$

Without loss of generality we can assume that

$$(\log p)^{1/2} \leqslant \varphi_p \leqslant 4(\log p)^{1/2}. \tag{2.4}$$

Eq. (2.4) is the usual reduction in the study of upper and lower classes and can be established in a standard fashion. Let

$$\varphi_p^* = (\varphi_p \vee (\log p)^{1/2}) \wedge 4(\log p)^{1/2}.$$

By (1.16) we have $1 \leqslant M_p \leqslant \text{const}\, \varphi_p^2$ and thus the contribution of those p in the sum in (2.3) where $\varphi_p \geqslant 4(\log p)^{1/2}$ is finite. Also, if $\varphi_p \geqslant 4(\log p)^{1/2}$ for all p, then φ belongs to the upper class by the LIL (1.6). Next we observe that if there are infinitely many p's for which $\varphi_p \leqslant (\log p)^{1/2}$, then the sum in (2.3) and the analogous sum with φ_p^* diverge. Indeed, if $\varphi_{\mathrm{L}} \leqslant (\log L)^{1/2}$ for some $L \geqslant 1$, then by the monotonicity of φ and of the function $x\mathrm{e}^{-x^2/2}$ we get that the Lth partial sum of the sum in (2.3) is at least

$$\sum_{p=1}^{L} \varphi_p^{-1} \mathrm{e}^{-\varphi_p^2/2} \geqslant L\varphi_{\mathrm{L}}^{-1} \mathrm{e}^{-\varphi_L^2/2} \geqslant L^{1/2} (\log L)^{-1/2}$$

and the same argument applies for the analogous sum with φ_p^*. The rest of the argument is the same as in the classical case, see e.g. in Feller [6, Lemma 1].

Applying Proposition 2.1 to $T = \mathcal{N} \cap I_{\kappa_p}$ gives

$$C_1(M) \leqslant \frac{\mathbf{P}\{\sup_{t \in \mathcal{N} \cap I_{\kappa_p}} W_t/\sqrt{t} > x\}}{\Psi(x) M(\mathcal{N} \cap I_{\kappa_p}, 1/x)} \leqslant C_2(M) \quad \text{for } x \geqslant C_3(M). \tag{2.5}$$

To any $x \geqslant C_3(M)$ and any increasing sequence $\mathcal{N}$ of positive integers, associate a set $\mathcal{N}_p(x) \subset \mathcal{N} \cap I_{\kappa_p}$, maximal for the relation

$$s, t \in \mathcal{N}_p(x), s \neq t \quad \Rightarrow \quad d(s,t) \geqslant 1/x.$$

Applying Proposition 2.1 to $T = \mathcal{N}_p(x)$ gives

$$C_1(M) \leqslant \frac{\mathbf{P}\{\sup_{t \in \mathcal{N}_p(x)} W_t/\sqrt{t} > x\}}{\Psi(x)\#(\mathcal{N}_p(x))} \leqslant C_2(M) \quad \text{for } x \geqslant C_3(M). \tag{2.6}$$

Thus there exists a constant $c(M)$ depending on M only, such that for any arbitrary increasing sequence $\mathcal{N}$ of positive integers we have

$$c(M)^{-1} \leqslant \frac{\mathbf{P}\{\sup_{t \in \mathcal{N} \cap I_{\kappa_p}} W_t/\sqrt{t} > x\}}{\mathbf{P}\{\sup_{t \in \mathcal{N}_p(x)} W_t/\sqrt{t} > x\}} \leqslant c(M) \quad \text{for } x \geqslant c(M)^{-1}. \tag{2.7}$$

Put, for any positive integer p,

$$\mathcal{A}_p^* = \left\{ \sup_{n \in \mathcal{N}_p(\varphi_p)} W_n/\sqrt{n} > \varphi_p \right\}, \quad \mathcal{A}_p = \left\{ \sup_{n \in \mathcal{N} \cap I_{\kappa_p}} W_n/\sqrt{n} > \varphi_p \right\}.$$

By the construction, $\#(\mathcal{N}_p(\varphi_p)) = M(\mathcal{N} \cap I_{\kappa_p}, 1/\varphi_p)$. In view of (2.5)–(2.7) we have

$$C_1(M) \leqslant \frac{\mathbf{P}(\mathcal{A}_p)}{\Psi(\varphi_p)\#(\mathcal{N}_p(\varphi_p))} \leqslant C_2(M) \tag{2.8}$$

and

$$c(M)^{-1} \leqslant \frac{\mathbf{P}(\mathcal{A}_p)}{\mathbf{P}(\mathcal{A}_p^*)} \leqslant c(M). \tag{2.9}$$

Hence if

$$\sum_{p=1}^{\infty} \Psi(\varphi_p) M(\mathcal{N} \cap I_{\kappa_p}, 1/\varphi_p) < \infty,$$

then $\sum P(\mathscr{A}_p) < \infty$ and thus $P(\mathscr{A}_p \text{ i.o.}) = 0$, implying $\varphi \in \mathscr{U}_{\mathscr{N}}(W)$. Assume now that

$$\sum_{p=1}^{\infty} \Psi(\varphi_p) M(\mathscr{N} \cap I_{\kappa_p}, 1/\varphi_p) = \infty.$$

Then

$$\sum_{p=1}^{\infty} \mathbf{P}(\mathscr{A}_p^*) = \infty.$$

By (1.16) we have

$$\#(\mathscr{N}_p(\varphi_p)) = M(\mathscr{N} \cap I_{\kappa_p}, 1/\varphi_p) \leqslant \text{Const}\, \varphi_p^2.$$

Therefore,

$$\Psi(\varphi_p) \leqslant \Psi(\varphi_p) M(\mathscr{N} \cap I_{\kappa_p}, 1/\varphi_p) \leqslant \text{Const}\, \varphi_p^2 \Psi(\varphi_p). \tag{2.10}$$

We estimate now the probabilities of the intersections $\mathscr{A}_p^* \cap \mathscr{A}_q^*$. The line of arguments we use is classical and follows closely the proof of Theorem 3.3 in Weber [18]. We shall need classical correlation inequalities for jointly Gaussian pairs of random variables, for which we refer to [18, Lemma 3.4, p. 78] for simplicity. Specifically, we need the fact that if U, V are jointly Gaussian r.v.'s with $EU = EV = 0$, $EU^2 = EV^2 = 1$, $EUV = r \geqslant 0$, then

$$P(\min(U, V) > t) \leqslant P(U > t) \Psi\left(t\sqrt{\frac{1-r}{1+r}}\right) \quad (t \geqslant 0)$$

and further if $\varepsilon > 0$, then for any $x > 0$, $y > 0$ with $rxy \leqslant \varepsilon$ we have

$$P(U > x, V > y) \leqslant c(\varepsilon) P(U > x) P(V > y),$$

where $\lim_{\varepsilon \to 0} c(\varepsilon) = 1$.

Let now some $0 < \alpha < (1 - M^{-1/2})/4$ and $0 < h < 1$ be fixed.

(a) If $p_0 \leqslant p < q \leqslant p + p^\alpha$, then using the correlation inequalities above we get

$$\begin{aligned}
\mathbf{P}(\mathscr{A}_p^* \cap \mathscr{A}_q^*) &\leqslant \sum_{n \in \mathscr{N}_p(\varphi_p)} \sum_{m \in \mathscr{N}_q(\varphi_q)} \mathbf{P}\left\{\frac{W_n}{\sqrt{n}} > \varphi_p, \frac{W_m}{\sqrt{m}} > \varphi_q\right\} \\
&\leqslant \sum_{n \in \mathscr{N}_p(\varphi_p)} \sum_{m \in \mathscr{N}_q(\varphi_q)} \mathbf{P}\left\{\frac{W_n}{\sqrt{n}} \wedge \frac{W_m}{\sqrt{m}} > \varphi_p\right\} \\
&\leqslant \sum_{n \in \mathscr{N}_p(\varphi_p)} \Psi(\varphi_p) \left\{ \sum_{m \in \mathscr{N}_q(\varphi_q)} \Psi\left[\varphi_p \left(\frac{1 - r(n,m)}{1 + r(n,m)}\right)^{1/2}\right] \right\},
\end{aligned} \tag{2.11}$$

where we put $r(n,m) = \mathbf{E}\, \frac{W_n}{\sqrt{n}} \frac{W_m}{\sqrt{m}}$. But,

$$\mathbf{E} \frac{W_n}{\sqrt{n}} \frac{W_m}{\sqrt{m}} \leqslant M^{-(q-p-1)/2} \leqslant M^{-1/2}$$

if $n \in \mathcal{N}_p(\varphi_p)$, $m \in \mathcal{N}_q(\varphi_q)$ and $q > p+1$. We may thus continue our estimates as follows:

$$\begin{aligned}\mathbf{P}(\mathscr{A}_p^* \cap \mathscr{A}_q^*) &\leqslant \text{const}\left\{\sum_{n \in \mathcal{N}_p(\varphi_p)} \Psi(\varphi_p)\right\}\varphi_q^2 \Psi\left[\varphi_p\left(\frac{1-M^{-1/2}}{1+M^{-1/2}}\right)^{1/2}\right] \\ &\leqslant \text{const}\, \mathbf{P}(\mathscr{A}_p^*)(\log p)p^{-(1-M^{-1/2})/4}\end{aligned}$$

for $q > p+1$, using (2.4), (2.8), (2.9), $q \leqslant 2p$ and $\Psi(x) \leqslant \exp(-x^2/2)$ for $x \geqslant 1$. If $q = p+1$, then trivially $\mathbf{P}(\mathscr{A}_p^* \cap \mathscr{A}_q^*) \leqslant \mathbf{P}(\mathscr{A}_p^*)$. Hence,

$$\begin{aligned}\sum_{p < q \leqslant p+p^\alpha} \mathbf{P}(\mathscr{A}_p^* \cap \mathscr{A}_q^*) &\leqslant \mathbf{P}(\mathscr{A}_p^*) + \text{const}\, \mathbf{P}(\mathscr{A}_p^*)(\log p)p^{\alpha-(1-M^{-1/2})/4} \\ &\leqslant (1+h)\mathbf{P}(\mathscr{A}_p^*), \end{aligned} \tag{2.12}$$

provided p_0 is sufficiently large, which we assume.

(b) Assume now $p_0 \leqslant p < p + p^\alpha < q$. Fix some $0 < \beta < \alpha$. Then, as it is easy to see, $q - q^\beta > p$ provided that p_0 is large enough. Thus,

$$\begin{aligned}&\sup\left[\varphi_p\varphi_q \mathbf{E}\left\{\frac{W_n}{\sqrt{n}}\frac{W_m}{\sqrt{m}}\right\}, n \in \mathcal{N}_p(\varphi_p), m \in \mathcal{N}_q(\varphi_q), p \geqslant p_0, q > p + p^\alpha\right] \\ &\quad\leqslant \text{const}\ \sup\left[(\log p)^{1/2}(\log q)^{1/2}M^{-(q-p-1)/2},\ p \geqslant p_0, q > p+p^\alpha\right] \\ &\quad\leqslant \text{const}\ \sup\left[(\log q)M^{-(q^\beta-1)/2},\ q \geqslant p_0\right].\end{aligned}$$

Thus using again the correlation inequalities above, we find that

$$\mathbf{P}\left\{\frac{W_n}{\sqrt{n}} > \varphi_p, \frac{W_m}{\sqrt{m}} > \varphi_q\right\} \leqslant (1+h)\Psi(\varphi_p)\Psi(\varphi_q),$$

provided that p_0 is large enough, which we assume. Hence,

$$\begin{aligned}\mathbf{P}(\mathscr{A}_p^* \cap \mathscr{A}_q^*) &\leqslant \sum_{n \in \mathcal{N}_p(\varphi_p)} \sum_{m \in \mathcal{N}_q(\varphi_q)} \mathbf{P}\left\{\frac{W_n}{\sqrt{n}} > \varphi_p, \frac{W_m}{\sqrt{m}} > \varphi_q\right\} \\ &\leqslant (1+h)\#(\mathcal{N}_p(\varphi_p))\#(\mathcal{N}_q(\varphi_q))\Psi(\varphi_p)\Psi(\varphi_q) \\ &\leqslant \text{const}\, \mathbf{P}(\mathscr{A}_p^*)\mathbf{P}(\mathscr{A}_q^*). \end{aligned} \tag{2.13}$$

By (2.12) and (2.13) we have

$$\sum_{p_0 \leqslant p < q \leqslant n} \mathbf{P}(\mathscr{A}_p^* \cap \mathscr{A}_q^*) \leqslant \text{const}\left(\sum_{p=1}^n \mathbf{P}(\mathscr{A}_p^*)\right)^2$$

and thus by invoking a well known version of the second Borel–Cantelli Lemma (see e.g. Spitzer [15, p. 317]) and using the 0–1 law, we deduce

$$\mathbf{P}\{\mathscr{A}_p^* \text{ i.o.}\} = 1.$$

Hence $\varphi \in \mathscr{L}_{\mathcal{N}}(W)$. The proof is now complete. $\square$

3. Upper and lower classes for partially observed sums of i.i.d. random variables

In the previous section we proved Theorem 1.1 in the Brownian case; we will now consider the general i.i.d. case.

Let $X = \{X_i, i \geqslant 1\}$ be a sequence of i.i.d. random variables satisfying $EX_1 = 0$, $EX_1^2 = 1$ and (1.5). By Theorem 2 of Einmahl [2] there exists, after suitably enlarging the probability space, a linear Brownian motion W such that

$$|S_n - W_n| = \mathrm{o}(n/\log\log n)^{1/2} \quad \text{a.s.} \tag{3.1}$$

This will permit to reduce the study of the classes $\mathscr{U}$ and $\mathscr{L}$ to the one of the Brownian motion. For, let

$$\varphi^{(1)}(n) = \varphi(n) + (\log\log n)^{-1/2}, \quad \varphi^{(2)}(n) = \varphi(n) - (\log\log n)^{-1/2} \tag{3.2}$$

and put $\Sigma(\varphi) = \sum_{p=1}^{\infty} \Psi(\hat{\varphi}_p) M(\mathscr{N} \cap I_{\kappa_p}, 1/\hat{\varphi}_p)$. We claim that

$$\Sigma(\varphi) < \infty \;\Rightarrow\; \Sigma(\varphi^{(2)}) < \infty,$$

$$\Sigma(\varphi) = \infty \;\Rightarrow\; \Sigma(\varphi^{(1)}) = \infty. \tag{3.3}$$

Indeed, by (3.2) we have

$$\hat{\varphi}_p^{(1)} = \hat{\varphi}_p + \mathrm{O}((\log p)^{-1/2}), \quad \hat{\varphi}_p^{(2)} = \hat{\varphi}_p + \mathrm{O}((\log p)^{-1/2})$$

and we can also assume, as noted above, that $\hat{\varphi}_p = \mathrm{O}((\log p)^{1/2})$. Hence by (2.1) the ratios $\Psi(\hat{\varphi}_p)/\Psi(\hat{\varphi}_p^{(1)})$ and $\Psi(\hat{\varphi}_p)/\Psi(\hat{\varphi}_p^{(2)})$ are bounded below and above by positive constants and thus using the monotonicity of the function $x \mapsto M(\mathscr{N} \cap I_{\kappa_p}, x)$ we get (3.3). This observation then implies, when combined with the already settled Brownian case of Theorem 1.1 and (3.1),

$$\Sigma(\varphi) < \infty \;\Rightarrow\; \Sigma(\varphi^{(2)}) < \infty \;\Rightarrow\; \varphi^{(2)} \in \mathscr{U}_{\mathscr{N}}(W) \;\Rightarrow\; \varphi \in \mathscr{U}_{\mathscr{N}}(X),$$

$$\Sigma(\varphi) = \infty \;\Rightarrow\; \Sigma(\varphi^{(1)}) = \infty \;\Rightarrow\; \varphi^{(1)} \in \mathscr{L}_{\mathscr{N}}(W) \;\Rightarrow\; \varphi \in \mathscr{L}_{\mathscr{N}}(X). \tag{3.4}$$

This completes the proof of Theorem 1.1.

4. Frequency results

Given an increasing sequence $\mathscr{N}$ of positive integers, Theorem 1.1 characterizes the upper and lower classes $\mathscr{U}_{\mathscr{N}}(X)$ and $\mathscr{L}_{\mathscr{N}}(X)$ for a large class of i.i.d. sequences $X = \{X_k,\ k \geqslant 1\}$. In this section we investigate the structure of the set

$$A = \{n \in \mathscr{N} : S_n(X) > \sqrt{n}\varphi(n)\} \tag{4.1}$$

for lower class functions $\varphi \in \mathscr{L}_{\mathscr{N}}(X)$. More precisely, we will investigate how frequently the inequality $S_n > \sqrt{n}\varphi(n)$ occurs along $\mathscr{N}$, i.e. how rapidly the sum

$$Z_n^{\mathscr{N}} = \sum_{k \leqslant n,\, k \in \mathscr{N}} I\{S_k > \sqrt{k}\varphi(k)\} \tag{4.2}$$

increases. In general, the behavior of $Z_n^{\mathcal{N}}$ is rather irregular; for example, if $\mathcal{N}$ is the whole sequence of positive integers and $\varphi(n) = c(2n \log \log n)^{1/2}$, $0<c<1$, then

$$\limsup_{j\to\infty} \frac{1}{j} Z_j^{\mathcal{N}} = 1 - \exp(-4(c^{-2}-1)), \quad \liminf_{j\to\infty} \frac{1}{j} Z_j^{\mathcal{N}} = 0 \quad \text{a.s.} \tag{4.3}$$

For the first relation of (4.3), see Strassen [16]; the second relation is a consequence of the proof. We thus see that the set A in (4.1) has no asymptotic density even in this simple case. We will show, however, that on an exponential time scale things behave quite nicely: the number of intervals $[M^k, M^{k+1})$ intersecting A has a simple asymptotic behavior. In fact, we have the following

Theorem 4.1. *Let $X = \{X_i, i\geqslant 1\}$ be an i.i.d. sequence satisfying $EX_1 = 0$, $EX_1^2 = 1$ and* (1.5). *Let $\mathcal{N}$ be an increasing sequence of positive integers and $\{\varphi(n), n\geqslant 1\}$ a non-decreasing sequence of positive reals tending to $+\infty$ and belonging to the lower class $\mathcal{L}_{\mathcal{N}}(X)$. Let*

$$\Upsilon_N = \#\{p\leqslant N : \exists n \in I_{\kappa_p} \cap \mathcal{N} : S_n > \sqrt{n}\varphi(n)\}. \tag{4.4}$$

Then we have

$$\Upsilon_N \asymp T_N \quad a.s., \tag{4.5}$$

where

$$T_N = \sum_{p\leqslant N} \frac{1}{\hat{\varphi}_p} \exp(-\hat{\varphi}_p^2/2) M(\mathcal{N} \cap I_{\kappa_p}, 1/\hat{\varphi}_p)$$

and $a_n \asymp b_n$ means $0< \liminf a_n/b_n \leqslant \limsup a_n/b_n < \infty$.

Theorem 4.1 contains the lower class part of Theorem 1.1, revealing the meaning of the sum in (1.11): its partial sums count (up to a constant factor) the number of exponential intervals $[M^k, M^{k+1})$ intersecting the set A in (4.1). The proof of the theorem will show that in the case when X_n are i.i.d. $N(0,1)$ r.v.'s and φ and $\mathcal{N}$ satisfy certain regularity conditions, one has the more precise asymptotics

$$\Upsilon_N = a_N + \mathrm{O}(a_N^{1/2} \log^b a_N) \quad \text{a.s.} \tag{4.6}$$

for any $b>\frac{3}{2}$, where

$$a_N = \sum_{p\leqslant N} \mathbf{P}\left(\sup_{n\in I_{\kappa_p}\cap\mathcal{N}} W(n)/\sqrt{n} \geqslant \hat{\varphi}_p \right).$$

However, as we do not have asymptotic estimates for a_N beyond $a_N \asymp T_N$, the fine behavior of Υ_N remains open. If $\mathcal{N} = \{n_k, k\geqslant 1\}$ has at most one element in each interval $[M^k, M^{k+1})$, there is no problem: in this case (4.6) reduces to

$$\#\{p\leqslant N : S_{n_p} > \sqrt{n_p}\varphi(p)\} = a_N^{1/2} + \mathrm{O}(a_N^{1/2} \log^b a_N) \quad \text{a.s.,}$$

where

$$a_N = \sum_{p \leqslant N} \Psi(\varphi(p))$$

and thus in this case we have a strong explicit asymptotics for the counting function.

The following consequence of Theorem 4.1 gives a frequency result in connection with the LIL (1.6).

Corollary 4.1. *Let* (X_n) *be an i.i.d. sequence satisfying* $EX_1 = 0$, $EX_1^2 = 1$ *and* (1.5). *Let* $\mathcal{N}$ *be an increasing sequence of positive integers and let* $\varphi^*(n)$ *be the canonic norming function in the LIL* (1.6) (*i.e.* $\varphi^*(n) = (2\log(p+2))^{1/2}$ *for* $n \in I_{\kappa_p} \cap \mathcal{N}$). *Let* $0 < c \leqslant 1$ *and*

$$\Upsilon_N^* = \#\{i \leqslant N : \exists n \in \mathcal{N} \cap [M^i, M^{i+1}) : S_n > c\sqrt{n}\varphi^*(n)\}$$

$$b_N = \#\{i \leqslant N : \mathcal{N} \cap [M^i, M^{i+1}) \neq \emptyset\}.$$

Then

$$\Upsilon_N^* \asymp \sum_{p \leqslant b_N} (\log p)^{-1/2} p^{-c^2} M(\mathcal{N} \cap I_{\kappa_p}, (\log p)^{-1/2}) \quad a.s. \tag{4.7}$$

Since the number $M(\mathcal{N} \cap I_{\kappa_p}, (\log p)^{-1/2})$ in (4.7) lies between 1 and $C \log p$ by (1.16), in the case $0 < c < 1$ we get

$$b_N^{1-c^2} (\log b_N)^{-1/2} \ll \Upsilon_N^* \ll b_N^{1-c^2} (\log b_N)^{1/2} \quad \text{a.s.},$$

where $x_n \ll y_n$ means that $\limsup |x_n/y_n| < \infty$. Consequently,

$$\lim_{N \to \infty} \frac{\log \Upsilon_N^*}{\log b_N} = 1 - c^2 \quad \text{a.s.}$$

establishing (1.8). In the case $c = 1$ the situation is more delicate and the behavior of Υ_N^* depends sensitively on the packing number $M(\mathcal{N} \cap I_{\kappa_p}, (\log p)^{-1/2})$ in (4.7). First we note that by (4.7) and the previous estimate for $M(\mathcal{N} \cap I_{\kappa_p}, (\log p)^{-1/2})$ we have

$$(\log b_N)^{1/2} \ll \Upsilon_N^* \ll (\log b_N)^{3/2} \quad \text{a.s.}$$

and consequently,

$$\frac{1}{2} \leqslant \liminf_{N \to \infty} \frac{\log \Upsilon_N^*}{\log \log b_N} \leqslant \limsup_{N \to \infty} \frac{\log \Upsilon_N^*}{\log \log b_N} \leqslant \frac{3}{2} \quad \text{a.s.}$$

The actual value of the lim inf and lim sup here depend on the order of magnitude of $M(\mathcal{N} \cap I_{\kappa_p}, (\log p)^{-1/2})$. For example, if

$$M(\mathcal{N} \cap I_{\kappa_p}, (\log p)^{-1/2}) \asymp (\log p)^{\alpha} \quad (0 < \alpha < 1),$$

then (4.7) yields

$$\Upsilon_N^* \asymp (\log b_N)^{\alpha + 1/2}$$

whence

$$\lim_{N\to\infty} \frac{\log \Upsilon_N^*}{\log \log b_N} = \alpha + 1/2.$$

As we pointed out earlier, the behavior of the counting function $Z_n^{\mathcal{N}}$ in (4.2) can be rather complicated for subexponentially growing $\mathcal{N}$. We will show, however, that replacing ordinary frequencies with logarithmic frequencies, a completely general strong law can be proved along any subsequence $\mathcal{N}$.

Theorem 4.2. *Let $X = \{X_i, i \geqslant 1\}$ be an i.i.d. sequence satisfying $EX_1 = 0$, $EX_1^2 = 1$ and $EX_1^2(\log_+|X_1|)^\alpha < \infty$ for some $\alpha > 2$. Let $\mathcal{N}$ be an increasing sequence of positive integers and $\{\varphi(n), n \geqslant 1\}$ a nondecreasing sequence of positive reals. Let*

$$D_M = \sum_{k \leqslant M,\, k \in \mathcal{N}} \frac{1}{k}\, \Psi(\varphi(k)).$$

Then for any $b > \frac{3}{2}$ we have

$$\sum_{k \leqslant M,\, k \in \mathcal{N}} \frac{1}{k} I\{S_k > \sqrt{k}\varphi(k)\} = D_M + \mathrm{O}(D_M^{1/2} \log^b D_M) \quad a.s. \tag{4.8}$$

In the case $D_M = \mathrm{O}(1)$ relation (4.8) reduces to

$$\sum_{k \leqslant M,\, k \in \mathcal{N}} \frac{1}{k}\, I\{S_k > \sqrt{k}\varphi(k)\} = \mathrm{O}(1) \quad \text{a.s.}$$

This is the case iff

$$\sum_{k \in \mathcal{N}} \frac{1}{k\varphi(k)} \mathrm{e}^{-\varphi(k)^2/2} < \infty$$

and if the last sum diverges and $\varphi(k) \to \infty$, then

$$D_M \sim \frac{1}{\sqrt{2\pi}} \sum_{k \leqslant M,\, k \in \mathcal{N}} \frac{1}{k\varphi(k)}\, \mathrm{e}^{-\varphi(k)^2/2}. \tag{4.9}$$

The sum on the right-hand side of (4.9) resembles the sum (1.3) in the Kolmogorov–Erdős–Petrovski test, but it has slightly smaller terms. For example, if $\varphi(n) = (2 \log \log n + c \log_3 n)^{1/2}$, then D_M remains bounded iff $c > 1$, while (1.3) converges iff $c > 3$.

If $\mathcal{N}$ is the whole sequence of integers and $\varphi(n) = c\,(2 \log \log n)^{1/2}$ $(0 < c < 1)$, then Theorem 4.2 yields

$$\lim_{N\to\infty} \frac{c(1-c^2)(4\pi \log \log N)^{\frac{1}{2}}}{(\log N)^{1-c^2}} \sum_{k=1}^{N} \frac{1}{k}\, I\{S_k > c\sqrt{2k \log \log k}\} = 1 \quad \text{a.s.} \tag{4.10}$$

and for $\varphi(n) = (2 \log \log n)^{1/2}$ we get

$$\lim_{N\to\infty} \sqrt{\pi}(\log \log N)^{-1/2} \sum_{k=1}^{N} \frac{1}{k}\, I\{S_k > \sqrt{2k \log \log k}\} = 1 \quad \text{a.s.} \tag{4.11}$$

Recall that by the almost sure central limit theorem (ASCLT) we have (see e.g. Lacey and Philipp [10])

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \frac{1}{k} I\{S_k > x\sqrt{k}\} = 1 - \Phi(x) \quad \text{a.s. for any } x.$$

Thus relations (4.10)–(4.11) can be considered the law of the iterated logarithm corresponding to the ASCLT.

As we pointed out above, the ordinary averages of the indicators $I(S_k > \sqrt{k}\varphi(k))$ along $\mathcal{N}$ behave irregularly if $\mathcal{N}$ is the whole sequence of integers. The following result shows that a similar irregularity holds for all subexponential sequences $\mathcal{N}$ satisfying minor regularity conditions. Let us recall that for exponentially growing sequences $\mathcal{N}$ the frequency behavior is quite regular (see the comments after Theorem 4.1).

Theorem 4.3. *Let* $X = \{X_i, i \geqslant 1\}$ *be an i.i.d. sequence satisfying* $EX_1 = 0$, $EX_1^2 = 1$ *and let* $\mathcal{N} = \{n_k, k \geqslant 1\}$ *be an increasing sequence of positive integers such that* $n_{k+1}/n_k \to 1$, n_k/k *is nondecreasing and* $(\log n_k)/k$ *is nonincreasing for* $k \geqslant k_0$. *Put* $\varphi(n) = c(2 \log\log n)^{1/2}$, $0 < c < 1$. *Then*

$$\limsup_{M\to\infty} \frac{1}{n_M} \sum_{k \leqslant M} (n_{k+1} - n_k) I\{S_{n_k} > \sqrt{n_k}\varphi(n_k)\} = 1 - \exp(-4(c^{-2} - 1)) \quad a.s.$$

and

$$\liminf_{M\to\infty} \frac{1}{n_M} \sum_{k \leqslant M} (n_{k+1} - n_k) I\{S_{n_k} > \sqrt{n_k}\varphi(n_k)\} = 0 \quad a.s.$$

Proof of Theorem 4.1. We will need the following:

Lemma 4.1. *Let* $(\xi_k, k \geqslant 1)$ *be nonnegative bounded r.v.'s and assume that there is a positive sequence* (m_k) *with* $\sum m_k = \infty$ *such that*

$$\mathbf{E}\left| \sum_{i \leqslant l \leqslant j} (\xi_l - \mathbf{E}\xi_l) \right|^2 \leqslant C \sum_{i \leqslant l \leqslant j} m_l \quad (0 \leqslant i \leqslant j < \infty).$$

Let $M_k = \sum_{i \leqslant k} m_i$. *Then for any* $b > \frac{3}{2}$ *we have*

$$\sum_{i \leqslant N} (\xi_i - \mathbf{E}\xi_i) = \mathrm{O}(M_N^{1/2} \log^b M_N) \quad a.s.$$

The proof of this result is identical with the proof of Theorem 3.2 in Weber [20]. For limit theorems of this type under mixing conditions, see Philipp [13].

Turning to the proof of Theorem 4.1, we first assume that X_n are i.i.d. $N(0,1)$ variables. Let H denote the set of integers $p \geqslant 1$ for which the increment of $\varphi^2(n)$ over $I_{\kappa_p} \cap \mathcal{N}$ is at least 1. Similarly to the argument in the proof of Theorem 1.1, discarding the sets $I_{\kappa_p} \cap \mathcal{N}$, $p \in H$ from the set $\mathcal{N}$ changes Υ_N and T_N by O(1), and thus without loss of generality we can assume that $\varphi^2(n)$ changes by at most 1 on every set $I_{\kappa_p} \cap \mathcal{N}$. Let φ_{L}, resp. φ_{U} denote the functions obtained by replacing φ by its smallest, resp. largest value in each interval $I_{\kappa_p} \cap \mathcal{N}$, and let $\Upsilon_N^{(\mathrm{L})}$, $\Upsilon_N^{(\mathrm{U})}$, $T_N^{(\mathrm{L})}$, $T_N^{(\mathrm{U})}$ denote the analogues of Υ_N, T_N when φ is replaced by φ_{L} and φ_{U}, respectively. Clearly, Υ_N can be bounded below and above by $\Upsilon_N^{(\mathrm{U})}$ and $\Upsilon_N^{(\mathrm{L})}$. Also, $\varphi_U^2 - \varphi_L^2 \leqslant 1$, and thus by $\varphi(n) \to \infty$ we get $\varphi_{\mathrm{U}}(n) \sim \varphi_{\mathrm{L}}(n)$ as $n \to \infty$. Hence using (2.2) we see that $T_N \asymp T_N^{(\mathrm{U})} \asymp T_N^{(\mathrm{L})}$. Thus it suffices to prove Theorem 4.1 for φ_{U} and φ_{L} instead of φ. In other words, without loss of generality we can assume that on each interval $I_{\kappa_p} \cap \mathcal{N}$ the function φ takes a constant value, which we will denote again by φ_p. As in the proof of Theorem 1.1, the contribution of terms with $\varphi_p \geqslant 4(\log p)^{1/2}$ in T_N and Υ_N is O(1), and thus without loss of generality we can assume that $\varphi_p \leqslant 4(\log p)^{1/2}$.

We next show that without loss of generality we can assume that for each $p \geqslant 1$, the set $\mathcal{N}$ contains at most $\log^2 p$ terms in I_{κ_p}. (This is similar to the reduction of the set $I_{\kappa_p} \cap \mathcal{N}$ to the set $\mathcal{N}_p(\varphi_p)$ in the proof of Theorem 1.1, but we need it now in a slightly different form.) Divide, for each $p \geqslant 1$, the interval $I_{\kappa_p} = [M^{\kappa_p}, M^{\kappa_p+1})$ to $[\log^2 p]$ intervals $J_{p,1}, \dots, J_{p,[\log^2 p]}$ of equal length and let $\mathcal{N}^*$ denote the sequence obtained from $\mathcal{N}$ by keeping the smallest element of $\mathcal{N}$ in each interval $J_{p,v}$ with $\mathcal{N} \cap J_{p,v} \neq \emptyset$. Let

$$D_{p,v} = \max_{i,j \in J_{p,v}} \left| \frac{W_i}{\sqrt{i}} - \frac{W_j}{\sqrt{j}} \right|,$$

$$H_{p,v} = \frac{W_{m_{p,v}^{(2)}}}{\sqrt{m_{p,v}^{(2)}}} - \frac{W_{m_{p,v}^{(1)}}}{\sqrt{m_{p,v}^{(1)}}},$$

$$D_p = \max_{1 \leqslant v \leqslant [\log^2 p]} D_{p,v},$$

where $m_{p,v}^{(1)}$, $m_{p,v}^{(2)}$ denote the left and right endpoints of $J_{p,v}$. We claim that

$$D_p = \mathrm{O}((\log p)^{-1/2}) \quad \text{a.s.,} \tag{4.12}$$

i.e. the fluctuation of $W_n/\sqrt{n}$ in $J_{p,v}$ is $\mathrm{O}((\log p)^{-1/2})$, uniformly in v. Given an Ornstein–Uhlenbeck process U, the variance of the increment $U(t) - U(s)$ equals $2(1 - \mathrm{e}^{-|t-s|/2}) \leqslant |t-s|$ for $s < t$ and consequently

$$\mathbf{P}(|U(t) - U(s)| \geqslant x) \leqslant A \mathrm{e}^{-x^2/(2|t-s|)} \quad (x > 0),$$

where A is an absolute constant. Thus using Móricz et al. [12, Theorem 2.2], it follows that for any interval $[a, b]$ and finite set $H \subset [a, b]$ we have

$$\mathbf{P}\left(\max_{s,t \in H} |U(t) - U(s)| \geqslant x \right) \leqslant A_1 \mathrm{e}^{-A_2 x^2/|b-a|} \quad (x > 0)$$

with absolute constants A_1, A_2. Now

$$m_{p,v}^{(1)},\ m_{p,v}^{(2)} \asymp M^{\kappa_p}, \quad m_{p,v}^{(2)} - m_{p,v}^{(1)} \asymp M^{\kappa_p}/\log^2 p$$

and thus

$$\log m_{p,v}^{(2)} - \log m_{p,v}^{(1)} \asymp m_{p,v}^{(2)}/m_{p,v}^{(1)} - 1 \asymp (\log p)^{-2}.$$

Hence for the increment $D_{p,v}$ defined above we have

$$P(D_{p,v} \geqslant C(\log p)^{-1/2}) \leqslant A_1 \exp\{-A_2 C^2 (\log p)^{-1}/(\log m_{p,v}^{(2)} - \log m_{p,v}^{(1)})\} \leqslant p^{-2}$$

for sufficiently large C and consequently

$$P(D_p \geqslant C(\log p)^{-1/2}) \leqslant \frac{\log^2 p}{p^2}$$

proving (4.12). Relation (4.12) shows that replacing $\mathcal{N}$ by $\mathcal{N}^*$ in

$$\Upsilon_N = \sum_{p \leqslant N} I\left\{ \sup_{n \in \mathcal{N} \cap I_{\kappa_p}} W_n/\sqrt{n} > \varphi_p \right\} \tag{4.13}$$

amounts to changing φ_p to $\varphi_p + \mathrm{O}((\log p)^{-1/2})$, but by (2.2), $\varphi_p \to \infty$ and $\varphi_p \leqslant 4(\log p)^{1/2}$, such a perturbation changes T_N to T'_N where $T'_N \asymp T_N$. Thus it suffices to prove Theorem 4.1 with $\mathcal{N}$ replaced by $\mathcal{N}^*$. In other words, we can assume, as claimed above, that for each $p \geqslant 1$, the set $\mathcal{N}$ contains at most $\log^2 p$ terms in I_{κ_p}.

Since the Ornstein–Uhlenbeck process $U(t) = \mathrm{e}^{-t/2} W(\mathrm{e}^t)$ is strongly mixing with an exponential rate (see Kolmogorov and Rozanov [9]), the events $\mathcal{A}_p$ in the proof of Theorem 1.1 satisfy

$$|P(\mathcal{A}_p \cap \mathcal{A}_q) - P(\mathcal{A}_p)P(\mathcal{A}_q)| \leqslant K\mathrm{e}^{-c(q-p)} \quad (p < q)$$

for some constants K and c. Thus for the α in the proof of Theorem 1.1 we have

$$\begin{aligned} &\sum_{\{m \leqslant p < q \leqslant n,\, q \geqslant p+p^\alpha\}} |P(\mathcal{A}_p \cap \mathcal{A}_q) - P(\mathcal{A}_p)P(\mathcal{A}_q)| \\ &\leqslant K \sum_{p=m}^{n} \sum_{\{p+p^\alpha \leqslant q \leqslant n\}} \mathrm{e}^{-c(q-p)} \\ &\leqslant K \sum_{p=m}^{n} \mathrm{e}^{-cp^\alpha} \sum_{j=0}^{\infty} \mathrm{e}^{-cj} \leqslant K_1 \sum_{p=m}^{n} \mathrm{e}^{-cp^\alpha} \leqslant K_2 m^{-2}. \end{aligned} \tag{4.14}$$

Since $\mathcal{N}$ contains at most $\log^2 p$ elements in each I_{κ_p}, we can repeat the proof of relation (2.12) in the proof of Theorem 1.1 to get

$$\sum_{p < q \leqslant p+p^\alpha} P(\mathcal{A}_p \cap \mathcal{A}_q) \leqslant \text{const}\, P(\mathcal{A}_p).$$

The same argument yields

$$\sum_{p < q \leqslant p+p^\alpha} P(\mathcal{A}_p)P(\mathcal{A}_q) \leqslant \text{const}\, P(\mathcal{A}_p)$$

(note that in (2.11), the last expression with $r(n,m) = 0$ is an upper bound for $\mathbf{P}(\mathcal{A}_p^* \cap \mathcal{A}_q^*)$) and thus

$$\sum_{p < q \leqslant p+p^\alpha} |P(\mathcal{A}_p \cap \mathcal{A}_q) - P(\mathcal{A}_p)P(\mathcal{A}_q)| \leqslant \text{const}\, P(\mathcal{A}_p). \tag{4.15}$$

From (4.14) and (4.15) it follows that

$$\sum_{m \leqslant p \leqslant q \leqslant n} |P(\mathscr{A}_p \cap \mathscr{A}_q) - P(\mathscr{A}_p)P(\mathscr{A}_q)| \leqslant \text{const} \sum_{p=m}^{n} (P(\mathscr{A}_p) + p^{-2})$$

and thus letting $b_N = \sum_{p \leqslant N} P(\mathscr{A}_p)$, Lemma 4.1 yields

$$\sum_{p \leqslant N} (I_{\mathscr{A}_p} - P(\mathscr{A}_p)) = \mathrm{O}(b_N^{1/2} \log^b b_N) \quad \text{a.s.}$$

for any $b > \frac{3}{2}$. This implies Theorem 4.1 in the Wiener case, since $b_N \asymp T_N$ by Proposition 2.1.

Finally, let (X_n) be i.i.d. random variables satisfying $EX_1 = 0$, $EX_1^2 = 1$ and (1.5). Applying Einmahl [2, Theorem 2] it follows that there exists a Wiener process W with

$$S_n - W_n = \mathrm{O}((n/\log\log n)^{1/2}) \quad \text{a.s.} \tag{4.16}$$

In the reduction steps of the previous proof we reduced the study of Υ_N to the case when $\varphi(n)$ takes a constant value φ_p on each set $I_{\kappa_p} \cap \mathscr{N}$ and φ_p satisfies $\varphi_p \leqslant 4(\log p)^{1/2}$. By (4.16), replacing W_n by S_n in (4.13) amounts to a perturbation of φ_p by $\mathrm{O}(\log p)^{-1/2})$ and we have seen earlier that such a perturbation changes T_N to T_N^*, where $T_N^* \asymp T_N$ a.s. Thus the approximation (4.16) reduces the general i.i.d. case to the Wiener case. This completes the proof of Theorem 4.1. □

Proof of Theorem 4.2. By Theorem 2 of Einmahl [2] there exists a Wiener process W satisfying

$$S_n - W_n = \mathrm{O}(\sqrt{n}(\log n)^{-\beta}) \quad \text{a.s.} \tag{4.17}$$

for some $\beta > 1$. By (4.17), replacing S_k by W_k in the indicator $I\{S_k > \sqrt{k}\varphi(k)\}$ is equivalent to changing $\varphi(k)$ to $\varphi(k) + \mathrm{O}((\log k)^{-\beta})$ and since $|\Psi'(x)| \leqslant 1$, the mean value theorem shows that by such a perturbation D_M changes at most

$$\mathrm{O}\left(\sum_{k \leqslant M} \frac{1}{k(\log k)^{\beta}}\right) = \mathrm{O}(1).$$

Thus relation (4.17) reduces Theorem 4.2 to the i.i.d. $N(0,1)$ case.

Let now X_n be i.i.d. $N(0,1)$ variables, i.e. $S_n = W_n$, where W is a Wiener process. Let $A_k = \{W_k > \sqrt{k}\varphi(k)\}$, $\eta_k = I(A_k) - P(A_k)$; we have to prove

$$\sum_{k \leqslant M,\, k \in \mathscr{N}} \frac{1}{k} I(A_k) = D_M + \mathrm{O}(D_M^{1/2} \log^b D_M) \quad \text{a.s.} \tag{4.18}$$

for any $b > \frac{3}{2}$. If $D_M = \mathrm{O}(1)$, then the left-hand side of (4.18) remains bounded in L_1 norm and thus by the monotone convergence theorem it is also a.s. bounded, i.e. (4.18) is valid. Assume now $D_M \to \infty$. By the Kolmogorov–Rozanov theorem (see [9, Theorem 1]) the correlation coefficient of $I(A_k)$ and $I(A_l)$ cannot exceed the correlation coefficient of $W(k)/\sqrt{k}$ and $W(l)/\sqrt{l}$, which is $\sqrt{k/l}$ for $k < l$.

This gives

$$
\begin{aligned}
|E(\eta_k\eta_l)| &= |P(A_kA_l) - P(A_k)P(A_l)| \\
&\leqslant \sqrt{\frac{k}{l}}\,P(A_k)^{1/2}P(A_l)^{1/2} \leqslant \sqrt{\frac{k}{l}}\,P(A_k) \quad (k<l)
\end{aligned}
$$

and thus

$$
\begin{aligned}
&E\left(\sum_{M\leqslant k\leqslant N,\, k\in\mathcal{N}} \frac{1}{k}\,\eta_k\right)^2 \\
&\quad\leqslant \sum_{M\leqslant k\leqslant N,\, k\in\mathcal{N}} \frac{1}{k^2}\,P(A_k) + 2 \sum_{M\leqslant k<l\leqslant N,\, k,\, l\in\mathcal{N}} \frac{1}{kl}\,\sqrt{\frac{k}{l}}\,P(A_k) \\
&\quad\leqslant \text{const} \sum_{M\leqslant k\leqslant N,\, k\in\mathcal{N}} \frac{1}{k}\,P(A_k).
\end{aligned}
$$

Hence applying Lemma 4.1 and using $D_M \to \infty$, we get (4.18), completing the proof of Theorem 4.2. □

Proof of Theorem 4.3. Since our proof is very similar to Strassen's proof of (4.3), we only sketch the argument. Let $n_0 = 0$, $S_0 = 0$ and let $X_M(t)$, $t \in [0,1]$ denote the function which is linear in the intervals $[n_k/n_M, n_{k+1}/n_M]$ $(0\leqslant k\leqslant M-1)$ and

$$
X_M(n_k/n_M) = (2n_M \log\log n_M)^{-1/2} S_{n_k} \quad (0\leqslant k\leqslant M).
$$

From $n_{k+1}/n_k \to 1$ and the results of Lifshits and Weber [11] it follows that the sequence X_M is relatively compact in the uniform topology of $C[0,1]$ and its cluster set is the Strassen set K, consisting of all absolutely continuous functions $x(t)$ in $(0,1)$ with $\int_0^1 x'(t)^2\,\mathrm{d}t\leqslant 1$. Moreover, by an argument in Strassen [16] we have for any $0<c<1$

$$
\limsup_{M\to\infty} \mu\{t \in [0,1] : X_M(t)\geqslant c\sqrt{t}\} = \gamma \quad \text{a.s.}, \tag{4.19}
$$

where μ denotes the Lebesgue measure and

$$
\gamma = \sup_{x\in K} \mu\{t \in [0,1] : x(t)\geqslant c\sqrt{t}\}.
$$

As Strassen also showed, $\gamma = 1 - \exp(-4(c^{-2} - 1))$. Put

$$
G_M(c) = \frac{1}{n_M} \sum_{0\leqslant k\leqslant M-1} (n_{k+1} - n_k) I\{S_{n_k} \geqslant \sqrt{n_k}\varphi(n_k)\}.
$$

From the assumptions made on n_k it follows that for any fixed $0<\alpha<1$ we have

$$
\log\log n_{[\alpha k]}/\log\log n_k \to 1, \quad n_{[\alpha k]}/n_k \leqslant \alpha.
$$

Thus letting $\varphi(n) = c\,(2\log\log n)^{1/2}$, $0<c<1$, we get

$$
X_M(t)/c\sqrt{t} \sim S_{n_k}/\sqrt{n_k}\varphi(n_k) \quad t \in [n_k/n_M, n_{k+1}/n_M)
$$

as $M \to \infty$, uniformly for $\alpha M \leqslant k \leqslant M$. Hence given any $\varepsilon > 0$, for sufficiently large M we have

$$\mu\{t \in [0,1] : X_M(t) \geqslant (c+\varepsilon)\sqrt{t}\} - \varepsilon \leqslant G_M(c) \leqslant \mu\{t \in [0,1] : X_M(t) \geqslant (c-\varepsilon)\sqrt{t}\} + \varepsilon$$

and thus (4.19) gives

$$\limsup_{M\to\infty} G_M(c) = \gamma \quad \text{a.s.}$$

proving the first statement of Theorem 4.3. The second statement can be proved similarly, using the fact that the function $x(t) \equiv 0$ is in the cluster set K. $\square$

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